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Bilinear discrete Painlevé equations

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Abstract. Based on the results of singularity confinement we derive bilinear expressions for the discrete Painlevé equations. In these cases where a bilinear expression is not sufficient we obtain trilinear or higher multilinear expressions. We show that the bilinear approach provides a natural framework for the derivation of Bäcklund and Schlesinger transforms for the discrete Painlevé equations.

1. Introduction

The Hirota bilinear formalism [1] has played a crucial role in the study of integrable nonlinear systems. Perfectly suitable for the derivation of multisoliton solutions, this approach has often been used for the investigation of the integrable (or not) character of evolution equations. Curiously, the bilinear formalism has not been particularly popular for the study of these equations that are the archetypes of integrable equations: the six transcendental equations of Painlevé and Gambier [2]. This is all the more curious since the solutions of these equations are meromorphic in the complex plane of the independent variable and should thus possess simple expressions in terms of ratios of entire functions. (This is precisely what the Hirota formalism does: introducing a dependent-variable transformation it allows the latter to be expressed in terms of the τ-functions which are assumed to be entire.) As a matter of fact this property of the Painlevé transcendents was pointed out by Painlevé himself [3]. However, the systematic application of the bilinear approach to the Painlevé equations had to wait till very recently: only in 1992 did Hietarinta and Kruskal [4] give the standard bilinear forms of the first five Painlevé transcendents (although this approach was initiated in the work of Okamoto, dating back to 1981 [5]). The causes of this neglect are not clear. Just as in the multidimensional case, the bilinear approach for Painlevé equations does lead to the systematic construction of particular solutions [6]. Moreover, the bilinear formalism provides a natural framework for the derivation of Bäcklund transforms [7]. These reasons should have encouraged a more active study of the bilinear Painlevé equations.

This paper does not focus on the bilinear continuous Painlevé equations but rather on their most interesting, newly discovered, next of kin: the discrete Painlevé equations. The latter are integrable difference equations which, at the continuous limit, go over to the continuous Painlevé. *The two share many common features* [8]. *One of these common* properties is the fact that they both possess bilinear expressions. Discrete Painlevé (dPs) equations have been recently identified in field theoretical models [9], although the very first instances of their occurrence are appreciably older [10]. A systematic approach for their derivation has been presented in [11] in the framework of the singularity confinment conjecture [12]. This integrability detector for discrete systems is based on the fact that the singularities which appear spontaneously in integrable discrete systems disappear after a few iterations. Singularity confinement is the discrete analogue of the Painlevé property for continuous systems (absence of movable critical singularities) [13]. Thus the use of singularity confinement for the derivation of the dPs establishes a perfect analogy with the continuous case and the Painlevé–Gambier derivation of the continuous Painlevé equations. As we shall see in what follows, the singularity structure will be a most valuable guide for the bilinearization of the dPs. Section 2 is devoted to precisely this question, while the rest of the paper deals systematically with the derivation of the bilinear expressions for the known dPs.

2. Tau-functions from singularity confinement

Before embarking upon the construction of the bilinear forms of dPs, let us examine the singularity structure of some simple cases and try to put it to use in the choice of the dependent variable transformation. The two cases that we will examine here are the standard $\mathrm{dP_I}$ and $\mathrm{dP_{II}}$, known under the forms

$$x_{n+1}+x_{n-1}=-x_n+\frac{z}{x_n}+a \tag{2.1}$$

$$x_{n+1}+x_{n-1}=\frac{zx_n+a}{1-x_n^2} \tag{2.2}$$

respectively. Here z is linear in the independent variable n, i.e. $z=\alpha n+\beta$. For the needs of the present paper a schematic singularity structure will suffice (the precise balancing can be found in [6, 8] and is not necessary here). In the case of $\mathrm{dP_I}$ we have a singularity whenever the x_n in the denominator happens to vanish. This has as a consequence that both x_{n+1} and x_{n+2} diverge, whereupon x_{n+3} vanishes again and x_{n+4} is finite (i.e. the singularity is indeed confined). Thus the singularity pattern is $\{0,\infty,\infty,0\}$. In the case of $\mathrm{dP_{II}}$ a singularity appears whenever x_n in the denominator takes the value $+1$ or -1. Thus we have two singularity patterns, which in this case turn out to be $\{-1,\infty,+1\}$ and $\{+1,\infty,-1\}$.

How can we use this information in order to express x in terms of τ-functions? Let us start with $\mathrm{dP_I}$. As will become clear in what follows there exists a relationship between the singularity patterns of a dP and the number of τ-functions necessary for its description. Thus in the case of $\mathrm{dP_I}$, which has a unique singularity pattern, it is enough to introduce just one τ-function. Since τ-functions are entire, x must be a ratio of products of such functions. Hence, let us assume that x_n contains a τ-function F_n in the numerator and that F_n passes through zero. Since x_{n+1} and x_{n+2} are infinite, the denominator of x must contain F_{n-1} and F_{n-2} (which ensures that F_n appears in the denominators of x_{n+1} and x_{n+2} respectively). Finally since x_{n+3} vanishes, x_n must contain F_{n-3} at the numerator. Thus, the expression for x, dictated by the singularity pattern, is

$$x_n=\frac{F_nF_{n-3}}{F_{n-1}F_{n-2}}. \tag{2.3}$$

As we shall see in the following sections, this expression suffices for the multilinearization (more precisely, trilinearization) of $\mathrm{dP_I}$. That the choice (2.3) is a reasonable one can also be seen through the continuous limit of this expression. We know, for $\mathrm{dP_I}$, that the continuous limit is obtained through $x = 1 + \epsilon^2 w$ at $\epsilon \to 0$. Implementing this limit on (2.3) we find $w = 2\partial_z^2 \log F$, a transformation that is at the base of the (continuous) Hirota bilinear formalism.

In the case of $\mathrm{dP_{II}}$ we have two singularity patterns, and so we expect two τ-functions to appear in the expression of x. Let us start with the pattern $\{-1, \infty, +1\}$. The diverging x may be related to a vanishing τ-function, say F, in the denominator. In order to ensure that x_{n-1} and x_{n+1} are respectively -1 and $+1$, we choose x_n in the form: $x_n = -1 + (F_{n+1}/F_n)p = 1 + (F_{n-1}/F_n)q$, where p, q must be expressed in terms of the second τ-function G. We turn now to the second pattern $\{+1, \infty, -1\}$ related to the vanishing of the τ-function G. We find, in this case, $x_n = 1 + (G_{n+1}/G_n)r = -1 + (G_{n-1}/G_n)s$, where r, s are expressed in terms of F. Combining the two expressions in terms of F and G we find, with the appropriate choice of gauge, the following simple expression for x

$$x_n = -1 + \frac{F_{n+1}G_{n-1}}{F_n G_n} = 1 - \frac{F_{n-1}G_{n+1}}{F_n G_n} \tag{2.4}$$

which satisfies both singularity patterns. Thanks to this particular choice of gauge the relative sign is such that the continuous limit of (2.4), obtained through $x = \epsilon w$, is $w = \partial_z \log(F/G)$, i.e. precisely the expected transformation in the case of $\mathrm{P_{II}}$. Expressions (2.4) can be written in a way that recalls the bilinear formalism for continuous systems. The Hirota D operator plays an important part here also. Starting with D, defined through its action on the dot-product $Df \cdot g = (\partial_x - \partial_{x'}) f(x)g(x')|_{x=x'} = f'(x)g(x) - f(x)g'(x)$, we introduce the shift operator e^D. Its action on a dot-product is: $\mathrm{e}^{\lambda D} f \cdot g = f(x+\lambda)g(x-\lambda)$ and thus $F_{n+1}G_{n-1}$ can be obtained simply as $\mathrm{e}^D F \cdot G$ where e^D operates on the discrete variable n. As we have shown in [14], where we have discussed in detail the integrability of trilinear equations, the bilinear D operators are the building blocks of the higher multilinear ones: one must just specify on which variables the operator acts. For example, the quantity $F_{n+3}G_{n-2}H_{n-1}$ can be simply written as $\mathrm{e}^{2D_{12}+D_{13}} F \cdot G \cdot H$.

3. Bilinear expressions for $\mathrm{dP_{II}}$ and related equations

We start our systematic construction of bilinear expressions for dPs with the second discrete Painlevé equation. As explained in the previous chapter, two τ-functions are needed here, related to the nonlinear variable through (2.4) and thus we expect $\mathrm{dP_{II}}$ to be given as a system of two bilinear relations. Equation (2.4) does indeed provide the first equation of the system. By eliminating the denominator $F_n G_n$ we obtain

$$F_{n+1}G_{n-1} + F_{n-1}G_{n+1} - 2F_n G_n = 0. \tag{3.1}$$

In order to obtain the second equation we rewrite $\mathrm{dP_{II}}$: $(x_{n+1}+x_{n-1})(1-x_n)(1+x_n) = zx_n + a$. We use the two possible definitions of x_n in terms of F, G in order to simplify the expressions $1 - x_n$ and $1 + x_n$. Next, we obtain two equations by using these two definitions for x_{n+1} combined with the alternate definition for x_{n-1}. We thus obtain

$$F_{n+2}F_{n-1}G_{n-1} - F_{n-2}F_{n+1}G_{n+1} = F_n^2 G_n(zx_n + a) \tag{3.2a}$$

$$G_{n-2}G_{n+1}F_{n+1} - G_{n+2}G_{n-1}F_{n-1} = G_n^2 F_n(zx_n + a). \tag{3.2b}$$

Finally, we add equation (3.2*a*) multiplied by G_{n+2} and (3.2*b*) multiplied by F_{n+2}. Up to the use of the upshift of (3.1), a factor $F_{n+1}G_{n+1}$ appears in both sides of the resulting expression. After simplification, the remaining equation is indeed bilinear:

$$F_{n+2}G_{n-2} - F_{n-2}G_{n+2} = z(F_{n+1}G_{n-1} - F_{n-1}G_{n+1}) + 2aF_n G_n \tag{3.3}$$

where a symmetric expression was used for x in the right-hand side, obtained as the arithmetic mean of the two right-hand sides of (2.4). Equations (3.1) and (3.3), taken together, are the bilinear form of $\mathrm{dP_{II}}$.

The calculation we just presented for $\mathrm{dP_{II}}$ can be easily extended to the $\mathrm{dP_{III}}$ equation we presented in [15]. We have shown there that the system:

$$x_{n+1}+x_n=\frac{zy_n+c}{b^2-y_n^2} \tag{3.4a}$$

$$y_n+y_{n-1}=\frac{\tilde{z}x_n+d}{a^2-x_n^2} \tag{3.4b}$$

(where $\tilde{z}$ is a half-downshifted z, i.e. $\tilde{z}=z-\alpha/2$) goes over to the standard form of $\mathrm{P_{III}}$ at the continuous limit. Equation (3.4) came about as an off-shoot of $\mathrm{dP_{II}}$. Indeed the singularity confinement criterion for an equation of the form (2.2) leads to

$$x_{n+1}+x_{n-1}=\frac{zx_n+a+\delta(-1)^n}{1-x^2}. \tag{3.5}$$

The $\delta(-1)^n$ can be absorbed in a proper redefinition of even and odd x's leading finally to the system (3.4). Thus, when the even–odd dependence is present, we have a $\mathrm{dP_{III}}$, while, when we suppress it (and this can be considered as a particular type of coalescence, typical of the discrete setting) we obtain $\mathrm{dP_{II}}$. The bilinear expression for (3.4) can be worked out along the same lines as for (2.2). Four τ-functions are needed here, related to x and y through

$$x_n=-a+\frac{H_nK_{n-1}}{F_nG_n}=a-\frac{H_{n-1}K_n}{F_nG_n} \tag{3.6a}$$

$$y_n=-b+\frac{F_{n+1}G_n}{H_nK_n}=b-\frac{F_nG_{n+1}}{H_nK_n}. \tag{3.6b}$$

Thus, the first two bilinear equations for $\mathrm{dP_{III}}$ are:

$$H_nK_{n-1}+H_{n-1}K_n=2aF_nG_n \tag{3.7a}$$

$$F_{n+1}G_n+F_nG_{n+1}=2bH_nK_n. \tag{3.7b}$$

Repeating the calculation along the very same lines as for $\mathrm{dP_{II}}$ we obtain finally the two supplementary equations:

$$H_{n+1}K_{n-1}-H_{n-1}K_{n+1}=a(z(F_{n+1}G_n-F_nG_{n+1})+2cH_nK_n) \tag{3.8a}$$

$$F_{n+1}G_{n-1}-F_{n-1}G_{n+1}=b(\tilde{z}(H_nK_{n-1}-H_{n-1}K_n)+2dF_nG_n) \tag{3.8b}$$

which, together with (3.7a, b), complete the bilinear expression of $\mathrm{dP_{III}}$, equation (3.4).

Another interesting equation, in the family of $\mathrm{dP_{II}}$, is equation $\mathrm{dP_{34}}$ that plays the role of the 'modified' $\mathrm{dP_{II}}$. The (awkward) name $\mathrm{dP_{34}}$ is chosen by analogy to the continuous case where equation (34) in the Painlevé–Gambier classification is related to $\mathrm{P_{II}}$ through a simple Miura transformation. In [16] we have given the form of $\mathrm{dP_{34}}$:

$$(y_{n+1}+y_n)(y_n+y_{n-1})=\frac{4y_n^2-m^2}{y_n+\tilde{z}/2}. \tag{3.9}$$

What is the singularity structure of $\mathrm{dP_{34}}$? An analysis of (3.9) yields the following pattern: if y_n is equal to $-\tilde{z}_n/2$ then y_{n+1} diverges, $y_{n+2}=-y_{n+1}$ and finally $y_{n+3}=-\tilde{z}_{n+3}/2$, whereupon y_{n+4} turns out to be finite. Thus the singularity pattern for the variable $y+\tilde{z}/2$

is $\{0, \infty, \infty, 0\}$ just as in the case of dP$_{\rm I}$ and, so, we expect the expression of y in terms of τ-functions to be just

$$y_n = -\tilde{z}/2 + \frac{\tau_n \tau_{n-3}}{\tau_{n-1}\tau_{n-2}}. \tag{3.10}$$

The multilinear expression for dP$_{34}$ can be found on the basis of this transformation. It turns out that the resulting equation is a hexalinear one that cannot be reduced further, unless subsidiary dependent variables are introduced. However, there exists a simpler expresssion for this multilinear equation. The starting point is the discrete Miura [17] that relates dP$_{34}$ to dP$_{\rm II}$:

$$y_n = -\tilde{z}/2 + (x_n + 1)(1 - x_{n+1}). \tag{3.11}$$

Expressing x in terms of the τ-functions F and G we find

$$y_n = -\tilde{z}/2 + \frac{G_{n+2}G_{n-1}}{G_{n+1}G_n} \tag{3.12}$$

i.e. (3.10) leads to (3.12) provided we take $\tau_n = G_{n+2}$. The existence of the Miura makes the bilinear expression of dP$_{34}$ straightforward. In fact it is exactly the same as the one for dP$_{\rm II}$, i.e. (3.1) and (3.3). The only difference lies in the definition of the nonlinear function which in the case of dP$_{\rm II}$ is given by (2.4) while for dP$_{34}$ it is just (3.10). In fact, the hexalinear equation for dP$_{34}$ can be obtained starting from (3.1), (3.3) and eliminating F. This construction has also the advantage of relating the parameter m of dP$_{34}$ to that of dP$_{\rm II}$ through $m^2 = (a + \alpha/2)^2$.

As we pointed out in the introduction the bilinear formalism provides the natural framework for the derivation of the auto-Bäcklund transforms for the dPs. We shall illustrate this point in the case of dP$_{\rm II}$. We have seen above that, starting from dP$_{\rm II}$ with τ-functions F and G, we can obtain, from the compatibility of (3.1)–(3.3), a dP$_{34}$ on G alone with parameter $m^2 = (a + \alpha/2)^2$. The important point here is that at the level of dP$_{34}$ only m^2 is fixed. We can thus change the sign of m without changing the equation dP$_{34}$. The fact that G satisfies dP$_{34}$ means precisely that the two equations (3.1) and (3.3) are compatible with a such $m = a + \alpha/2$. But then two equations of the same form as (3.1), (3.3) with the very same G but with F replaced by some other τ-function H and a replaced by b will also be compatible provided that $-m = b + \alpha/2$. This corresponds to dP$_{\rm II}$ for the ordered pair (H, G) with parameter $b = -m - \alpha/2$ or, by inspection of the symmetry properties of (3.1)–(3.3) upon interchanging H and G, to a dP$_{\rm II}$ for the ordered pair (G, H) with parameter $a' \equiv -b = m + \alpha/2 = a + \alpha$. Thus starting from (F, G) with parameter a we can construct a pair (G, H) with parameter $a + \alpha$. This is precisely the auto-Bäcklund of dP$_{\rm II}$. The details of the construction of the function H can be found in [18]. We give here just the result of this calculation:

$$H_n F_{n+1} = G_{n+1}G_n(z + a + \alpha) - 2G_{n+2}G_{n-1} \tag{3.13a}$$

$$H_{n+1} F_n = G_{n+1}G_n(z - a) - 2G_{n+2}G_{n-1} \tag{3.13b}$$

or combining (3.13*a*, *b*):

$$H_n F_{n+1} - H_{n+1} F_n = (2a + \alpha) G_{n+1} G_n. \tag{3.14}$$

Thus starting from a given (F, G) at parameter a we can construct H ((3.13a) suffices), and iterating further (also backwards) we obtain the solution at any $a + n\alpha$, with n a relative integer.

The method used above for the construction of the auto-Bäcklund is quite general indeed. The important ingredient is that the bilinear equation for a given dP is, in fact, linear in each of the τ-functions. Thus we can eliminate one of them systematically and obtain a compatibility condition for the other τ-function. The resulting equation can be in general multilinear but the important point is that it depends on the parameter of the initial dP in a way that is invariant under some discrete transformations (in this case $m \leftrightarrow -m$). It is precisely the use of this discrete symmetry that allows us to establish the auto-Bäcklund transform as we have seen in the case of $\mathrm{dP_{II}}$.

4. Multilinear expressions for the various $\mathrm{dP_I}$s

An interesting property of the discrete Painlevé equations is that they possess several alternate forms, i.e. there exist *more than one* discrete systems (presumably integrable) which, at the continuous limit, go over to the *same* continuous Painlevé equation. This is all the more true for $\mathrm{dP_I}$.

In this section we will present the multilinearization of some of them, limiting ourselves to the simplest cases. Let us start with the 'standard' $\mathrm{dP_I}$ given by

$$x_{n+1} + x_n + x_{n-1} = z_n/x_n + a. \tag{4.1}$$

As we have seen in section 2 , the proper dependent variable transform, dictated by the singularity patterns of $\mathrm{dP_I}$, $\{0, \infty, \infty, 0\}$, is

$$x = \frac{F_{n+1}F_{n-2}}{F_n F_{n-1}}. \tag{4.2}$$

Instead of substituting into (4.1) we use the discrete derivative of the latter:

$$x_{n+2} - x_{n-1} = z_{n+1}/x_{n+1} - z_n/x_n.$$

By reducing to a common denominator we obtain the following trilinear form:

$$F_{n+3}F_{n-1}F_{n-2} - F_{n-3}F_{n+1}F_{n+2} = z_{n+1}F_{n+1}^2 F_{n-2} - z_n F_{n-1}^2 F_{n+2} \tag{4.3}$$

that cannot be reduced further. So the standard $\mathrm{dP_I}$ does not possess a bilinear form but rather a trilinear one.

Another simple form of $\mathrm{dP_I}$ is given by [12]

$$x_{n+1} + x_{n-1} = z_n/x_n + a. \tag{4.4}$$

Here, the singularity pattern is $\{0, \infty, a, \infty, 0\}$. This pattern suggests the transformation:

$$x = \frac{F_{n+2}F_{n-2}}{F_{n+1}F_{n-1}}. \tag{4.5}$$

Substituting back into (4.4) we obtain readily

$$F_{n+3}F_{n-1}F_{n-2} - F_{n-3}F_{n+1}F_{n+2} = z_n F_{n+1}F_n F_{n-1} + aF_{n+2}F_n F_{n-2} \tag{4.6}$$

again a trilinear form.

Another form of $\mathrm{dP_I}$ is known [16]:

$$x_{n+1} + x_{n-1} = z_n/x_n + a/x_n^2 \tag{4.7}$$

with singularity pattern $\{0, \infty, 0\}$. Moreover, when one examines the structure of the singularities more closely we find that they behave as $\epsilon, 1/\epsilon^2, \epsilon$ for $\epsilon \to 0$. This, in turn, leads to

$$x = \frac{F_{n+1}F_{n-1}}{F_n^2} \tag{4.8}$$

and upon substitution we obtain still another new trilinear expression:

$$F_{n+2}F_{n-1}^2 - F_{n-2}F_{n+1}^2 = z_n F_{n+1}F_n F_{n-1} + aF_n^3. \tag{4.9}$$

The last form of $\mathrm{dP_I}$ we shall treat here is not as well known as the other ones. It can be obtained [19] as a particular limit of $\mathrm{dP_{III}}$ (5.1) and reads

$$x_{n+1}x_{n-1} = z_n/x_n + a/x_n^2. \tag{4.10}$$

where, contrary to all the previous cases, here z_n is not an affine function of n, but rather $z_n = z_0\lambda^n$. (More precisely, the singularity confinement condition is just $z_{n+2}z_{n-2} = z_n^2$.) The singularity pattern of this equation is $\{0, \infty, 0\}$ as in the case of (4.7) and thus the same transformation (4.8) should be used here. But here a full F^2 factor drops out and we obtain

$$F_{n+2}F_{n-2} = z_n F_{n+1}F_{n-1} + aF_n^2. \tag{4.11}$$

Thus the $\mathrm{dP_I}$ (4.10) has a bilinear expression.

5. The third discrete Painlevé transcendent

The third discrete Painlevé transcendent occupies a unique position: it is, in fact, the only equation among the 'standard' dPs whose form was obtained by singularity confinement [11] prior to the obtention of its Lax pair [20] (since the Lax pair for the two remaining dPs, namely the fourth and fifth, are still unknown). The form of $\mathrm{dP_{III}}$ we are going to work with in what follows is

$$x_{n+1}x_{n-1} = \frac{ab(x_n - c)(x_n - d)}{(x_n - a)(x_n - b)} \tag{5.1}$$

where a, b are constants and c, d of the form $c = c_0\lambda^n$, $d = d_0\lambda^n$. Two singularity patterns exist for (5.1). Either x_n first takes the value a, in which case we have $\{a, \infty, b\}$ and the singularity is confined, or we start with b: $\{b, \infty, a\}$. This suggests the introduction of two τ-functions F, G in the form

$$x_n = a\left(1 + \frac{F_{n+1}G_{n-1}}{F_nG_n}\right) = b\left(1 + \frac{F_{n-1}G_{n+1}}{F_nG_n}\right). \tag{5.2}$$

The first bilinear equation is readily obtained from (5.2):

$$aF_{n+1}G_{n-1} - bF_{n-1}G_{n+1} + (a-b)F_nG_n = 0. \tag{5.3}$$

In order to obtain the second equation, we substitute back into (5.1) using the two different definitions for x in x_{n+1} and x_{n-1}. We thus obtain

$$ab(F_{n+1}G_{n+1} + F_{n+2}G_n)(F_{n-1}G_{n-1} + F_{n-2}G_n) = F_n^2G_n^2(x-c)(x-d). \tag{5.4}$$

Expanding the right-hand side, we remark that a term $abF_{n-1}G_{n-1}F_{n+1}G_{n+1}$ appears that cancels the one on the left-hand side, whereupon one can simplify by G_n and obtain

$$F_{n+2}F_{n-1}G_{n-1} + F_{n-2}F_{n+1}G_{n+1} + F_{n+2}F_{n-2}G_n = \frac{1}{ab}F_n\Phi \tag{5.5}$$

where

$$\Phi = (a-c)(b-d)F_nG_n + b(a-c)F_{n-1}G_{n+1} + a(b-d)F_{n+1}G_{n-1}. \tag{5.6}$$

Using the alternate definition for x in x_{n+1} and x_{n-1} we obtain an equation similar to (5.5) with $F \leftrightarrow G$ but the same Φ:

$$G_{n+2}F_{n-1}G_{n-1} + G_{n-2}F_{n+1}G_{n+1} + G_{n+2}G_{n-2}F_n = \frac{1}{ab}G_n\Phi. \tag{5.7}$$

Finally, we multiply (5.5) by G_{n+2}, (5.7) by F_{n+2} and subtract, we use (5.3) in order to simplify the result and obtain

$$aF_{n+2}G_{n-2} - bG_{n+2}F_{n-2} = \left(\frac{1}{a} - \frac{1}{b}\right)\Phi \tag{5.8}$$

with Φ given by equation (5.6). Thus $\mathrm{dP_{III}}$ indeed possesses a bilinear form.

6. Bilinear expressions for $\mathrm{dP_{IV}}$ and $\mathrm{dP_{V}}$

In order to study the fourth discrete Painlevé equation, we use its symmetric form introduced in [21]:

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{(x_n + \alpha + \beta)(x_n + \alpha - \beta)(x_n - \alpha + \beta)(x_n - \alpha - \beta)}{(x_n + z_n + \gamma)(x_n + z_n - \gamma)}. \tag{6.1}$$

Two singularity patterns are easily obtainable: x_{n-1} goes through one root of the denominator, x_n diverges and x_{n+1} goes through the other root, whereupon the singularity becomes confined. Thus we have two patterns $\{-z_{n-1} - \gamma, \infty, -z_{n+1} + \gamma\}$ and $\{-z_{n-1} + \gamma, \infty, -z_{n+1} - \gamma\}$. This suggests the ansatz:

$$x_n = -z_n - \gamma + \frac{F_{n+1}G_{n-1}}{F_nG_n} = -z_n + \gamma + \frac{F_{n-1}G_{n+1}}{F_nG_n}. \tag{6.2}$$

However, another 'potential' singularity exists for this equation. If $x_n = \pm(\alpha \pm \beta)$ then either x_{n-1} or x_{n+1} must be the opposite of x_n. But, upon closer study, this does *not* lead to a singularity. The precise pattern is $\{\pm(\alpha \pm \beta), \mp(\alpha \pm \beta)\}$, i.e. perfectly regular. Still this 'potential' singularity can be put to profit in order to introduce auxiliary τ-functions that will make possible the bilinearization of $\mathrm{dP_{IV}}$. We thus introduce

$$x_n = \alpha + \beta + \frac{H_nK_{n-1}}{F_nG_n} = -\alpha - \beta + \frac{H_{n-1}K_n}{F_nG_n} \tag{6.3a}$$

$$x_n = \alpha - \beta + \frac{M_nN_{n-1}}{F_nG_n} = -\alpha + \beta + \frac{M_{n-1}N_n}{F_nG_n}. \tag{6.3b}$$

Notice that there exists some asymmetry in the relations (6.3). To preserve the symmetry, we should have introduced a lattice shifted by a half-unit writing $H_{n+1/2}K_{n-1/2}$ instead of H_nK_{n-1}, but this 'half-lattice' notation would have been awkward in the long run.

Equations (6.2) and (6.3) provide five equations for the six τ-functions so only one remains to be obtained. Substituting in (6.1) we obtain

$$\begin{aligned}(K_{n-1}F_{n+1}G_{n+1} + K_{n+1}F_nG_n)(H_nF_{n-1}G_{n-1} + H_{n-2}F_nG_n)\\ = H_{n-1}K_{n+1}(x_n + \alpha - \beta)F_nG_n(x_n - \alpha + \beta)F_nG_n\end{aligned} \tag{6.4a}$$

and with the other choice for x_n:

$$\begin{aligned}(H_{n-1}F_{n+1}G_{n+1} + H_{n+1}F_nG_n)(K_nF_{n-1}G_{n-1} + K_{n-2}F_nG_n)\\ = H_{n+1}K_{n-1}(x_n + \alpha - \beta)F_nG_n(x_n - \alpha + \beta)F_nG_n.\end{aligned} \tag{6.4b}$$

Taking the difference of (6.4a) and (6.4b) we obtain an expression which can be simplified through the use of (6.3a). A factor F_nG_n drops out and we find

$$\begin{aligned}F_nG_n(K_{n+1}H_{n-2} - H_{n+1}K_{n-2}) + (\alpha + \beta)F_{n+1}G_{n+1}F_{n-1}G_{n-1}\\ = 2(\alpha + \beta)(x + \alpha - \beta)F_nG_n(x - \alpha + \beta)F_nG_n.\end{aligned} \tag{6.5}$$

From (6.2) we express the right-hand side in terms of F_n and G_n only and we remark that a further factor F_nG_n drops out. Finally we find a bilinear equation:

$$K_{n+1}H_{n-2} - H_{n+1}K_{n-2} = 2(\alpha+\beta)((\beta-\alpha+\gamma-z)F_{n+1}G_{n-1} + (\alpha-\beta-\gamma-z) \times F_{n-1}G_{n+1} + (\alpha-\beta-\gamma-z)(\beta-\alpha+\gamma-z)F_nG_n) \tag{6.6}$$

where z stands for z_n, or by symmetrizing further the right-hand side:

$$K_{n+1}H_{n-2} - H_{n+1}K_{n-2} = 2(\alpha+\beta)((\gamma^2+z^2-(\beta-\alpha)^2)F_nG_n - z \times(F_{n+1}G_{n-1} + F_{n-1}G_{n+1})) \tag{6.7}$$

which completes the bilinearization of $\mathrm{dP_{IV}}$.

At this point we can remark that only four of the six τ-functions have been used in the final equation (6.7). This suggests two possibilities. The first is to drop the two equations (6.3*b*), containing M and N, and have thus a system of four bilinear equations for $\mathrm{dP_{IV}}$. The second is to use M and N in order to make (6.7) more symmetric. This can be done by dividing (6.7) by $\alpha+\beta$ and subtracting the equation obtained by replacing H by M, K by N and changing the sign of β, resulting in

$$\frac{K_{n+1}H_{n-2} - H_{n+1}K_{n-2}}{\alpha+\beta} - \frac{N_{n+1}M_{n-2} - M_{n+1}N_{n-2}}{\alpha-\beta} = 8\alpha\beta F_nG_n. \tag{6.8}$$

In order to bilinearize $\mathrm{dP_V}$ we will start from the form

$$(x_nx_{n+1}-1)(x_nx_{n-1}-1) = \frac{p_nq_n(x_n-u)(x_n-1/u)(x_n-v)(x_n-1/v)}{(x_n-p_n)(x_n-q_n)} \tag{6.9}$$

where $p_n = p_0\lambda^n$, $q_n = q_0\lambda^n$, and u, v are constants. The more traditional form of $\mathrm{dP_V}$, found in [11], can be obtained from (6.9) by taking $y_n = \frac{1}{2}(x_n+1)$ whereupon, after some manipulations, one gets an equation the left-hand side of which is $(2y_n-1)y_{n+1}y_{n-1} - y_n(y_{n+1}+y_{n-1})$. The singularity patterns of (6.9) are easily obtained. Two genuinely singular sequences exist: $\{p_{n-1}, \infty, q_{n+1}\}$ and $\{q_{n-1}, \infty, p_{n+1}\}$. Moreover, a potential singularity exists when x_n takes a value that cancels the right-hand side of (6.9). Then, either x_{n+1} or x_{n-1} must be the inverse of x_n. This does not lead, however, to a singularity and the four patterns are: $\{u, 1/u\}$, $\{1/u, u\}$, $\{v, 1/v\}$, $\{1/v, v\}$. These singularity patterns suggest the introduction of six τ-functions in the following way:

$$x_n = p_n + \frac{F_{n+1}G_{n-1}}{F_nG_n} = q_n + \frac{F_{n-1}G_{n+1}}{F_nG_n} \tag{6.10}$$

$$x_n = u + \frac{H_nK_{n-1}}{F_nG_n} = \frac{1}{u} + \frac{H_{n-1}K_n}{F_nG_n} = v + \frac{M_nN_{n-1}}{F_nG_n} = \frac{1}{v} + \frac{M_{n-1}N_n}{F_nG_n}. \tag{6.11}$$

As in the case of $\mathrm{dP_{IV}}$, the asymmetry in the definitions of H, K, M and N could be removed through the introduction of a 'half-lattice' notation. Equations (6.10) and (6.11) provide five relations between the six τ-functions and so one more remains to be obtained. We shall not here go into the details of this calculation. It follows closely our derivation for $\mathrm{dP_{IV}}$. So we only give below the final expression, similar to (6.8), that reads:

$$\left(u-\frac{1}{u}\right)^{-1}\left(\frac{1}{u}H_{n+1}K_{n-2} - uK_{n+1}H_{n-2}\right) - \left(v-\frac{1}{v}\right)^{-1}\left(\frac{1}{v}M_{n+1}N_{n-2} - vN_{n+1}M_{n-2}\right) = \left(u+\frac{1}{u}-v-\frac{1}{v}\right)F_nG_n. \tag{6.12}$$

This completes the bilinearization of $\mathrm{dP_V}$. In contrast to the case of $\mathrm{dP_{IV}}$, we were not able to find a bilinearization in terms of four τ-functions only, similar to (6.7).

7. Discussion and outlook

A natural question that comes to the mind when one sees the results we have presented in the previous sections is 'Where is dP_{VI}?'. It is a fact that we have not yet obtained the discrete analogue of P_{VI} either in nonlinear or bilinear form. As far as the latter is concerned let us point out that the bilinear form of P_{VI} has not yet been obtained in the continuous case either. In fact, the whole approach of Kruskal and Hietarinta [4] was based on the existence for P_{III}, P_V of independent variable transformations that made it possible to absorb the terms depending linearly on the first derivative of the dependent variable. Such a transformation is not known for P_{VI} and this has been a stumbling block for its bilinearization. In the discrete case the situation is even worse. While one can show that the autonomous form of dP_{VI} coincides with the general symmetric Quispel mapping, no brute-force deautonomization can be performed: the bulk of the calculations is prohibitive. There also exists another indication that dP_{VI} may be more complicated than we initially thought. In [22] we have obtained just four consistent quantization schemes corresponding to the families of $dP_{I/II}$, dP_{III}, dP_{IV} and dP_V. This means that not all dPs or, if we limit ourselves to the autonomous case, not *all* Quispel mappings, can be quantized. Thus if dP_{VI} was given simply by a deautonomization of the autonomous form we have obtained it would not be quantizable. This is not a serious argument, of course, but allows us to indulge in some speculation. As we have shown in [15], while dP_{III} has a 'standard', symmetric form, there also exists an asymmetric form of dP_{III} that can be written as a system of two mappings belonging to the dP_{II} family. This is possible because of the extra freedom we often encounter in the discrete case where some coefficients have an even–odd dependence. Although the asymmetric form of dP_V has not yet been worked out, it may turn out that this technique works for dP_V and also for dP_{VI}. In this case the latter would have the form of two coupled 2-point mappings of the dP_V family. Still, we must admit that this is just speculation and no tangible results exist to date.

Another important question concerns the alternate dPs. As is well known by now, discrete P's exist in several forms. For example, in section 3, we have presented four different forms of dP_I and many more exist [19]. The classification problem of the dPs is completely open and the profusion of alternate forms makes it particularly interesting. We hope that the bilinear formalism will provide a useful tool for the investigation of this problem.

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